Lecture — 34.

Line Integral

Line Integral w.r.t. Arc Length

Let the parametric representation of a curve C be

C : $x = x(t)$, $y = y(t)$, $z = z(t)$; $a \le t \le b$

Then, we define the line integral of f over C w.r.t. arc length s as:

$$I = \int_C f(x,y,z)\, ds = \int_{t=a}^{b} \left\{ f[x(t), y(t), z(t)] \frac{ds}{dt} \right\} dt$$

$$\Rightarrow \ast \quad \boxed{I = \int_{t=a}^{b} f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt}$$

←——×——→

Ex: = 15.4

Evaluate $\int_C f(x,y)\, ds$ or $\int_C f(x,y,z)\, ds$

(1). $\boxed{f(x,y,z) = (2x + 3y), \quad C: x = t, y = 2t, z = 3t, 0 \le t \le 3}$

Here C : $x = t$, $y = 2t$, $z = 3t$; $0 \le t \le 3$

$$\Rightarrow \frac{dx}{dt} = 1 \;\Big|\; \frac{dy}{dt} = 2 \;\Big|\; \frac{dz}{dt} = 3$$

Also $f(x(t), y(t), z(t)) = 2(t) + 3(2t) = 8t$

$$\therefore I = \int_C f(x,y,z)\, ds = \int_C \left(f(t) \cdot \frac{ds}{dt}\right) dt$$

$$I = \int_{t=0}^{3} f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$\Rightarrow I = \int_{t=0}^{3} 8t \cdot \sqrt{(1)^2 + (2)^2 + (3)^2}\, dt$$

$$= 8 \int_{t=0}^{3} t \cdot \sqrt{14}\, dt = 8\sqrt{14} \left[\frac{t^2}{2}\right]_0^3$$

$$= \cancel{8}^{4} \left[\frac{9}{\cancel{2}} - 0\right] \sqrt{14} \Rightarrow \boxed{I = 36\sqrt{14}} \quad \text{Ans}$$

(2) $\boxed{f = x^2 y, \quad C: x = 3\cos t, \; y = 3\sin t, \quad 0 \le t \le \frac{\pi}{2}}$

Here $C: x = 3\cos t, \quad y = 3\sin t, \quad 0 \le t \le \frac{\pi}{2}$

$$\Rightarrow \frac{dx}{dt} = -3\sin t \quad \Big| \quad \frac{dy}{dt} = 3\cos t$$

$$f(t) = x^2 y = (3\cos t)^2 (3\sin t) = 27 \cdot \cos^2 t \sin t$$

$$\therefore I = \int_C f(x,y)\, ds = \int_{t=0}^{\pi/2} \left[f(x(t), y(t)) \cdot \frac{ds}{dt}\right] dt$$

$$\Rightarrow I = \int_{t=0}^{\pi/2} f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_{t=0}^{\pi/2} (27 \cdot \cos^2 t \sin t) \sqrt{9\sin^2 t + 9\cos^2 t}\, dt$$

$$= 27 \times 3 \int_{t=0}^{\pi/2} \cos^2 t \cdot \sin t\, dt$$

$$= -\cancel{81} \cdot \left[\frac{\cos^3 t}{\cancel{3}}\right]_0^{\pi/2} = -27\left[\cos^3\frac{\pi}{2} - \cos^3 0\right] = -27[0-1]$$

$$\Rightarrow \boxed{I = 27} \quad \text{Ans}$$

Line Integral of Vector Field (*Work Done) (III).

Let $C: \vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$; $a \le t \le b$

Write $\vec{V}(t) = V_1(t)\hat{i} + V_2(t)\hat{j} + V_3(t)\hat{k}$

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Write $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

$$I = \int_C \vec{V}\cdot d\vec{r} = \int_{t=a}^{b}\left[\vec{V}(t)\cdot\frac{d\vec{r}}{dt}\right]dt$$

$$\Rightarrow I = \int_{t=a}^{b}\left[(V_1\hat{i} + V_2\hat{j} + V_3\hat{k})\cdot\left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}\right)\right]dt$$

$$* \Rightarrow \boxed{I = \int_{t=a}^{b}\left(V_1\frac{dx}{dt} + V_2\frac{dy}{dt} + V_3\frac{dz}{dt}\right)dt}$$

* This is also called as Work done.

Evaluate line integral $\int_C \vec{V}\cdot d\vec{r}$

(1) $\boxed{\vec{V} = (xy\hat{i} + y^2\hat{j} + e^z\hat{k}),\ C: x = t^2, y = 2t, z = t, 0 \le t \le 1}$

Sol: $C: \vec{r} = (x\hat{i} + y\hat{j} + z\hat{k}) = t^2\hat{i} + 2t\hat{j} + t\hat{k}$

$$\Rightarrow \frac{dx}{dt} = 2t,\quad \frac{dy}{dt} = 2,\quad \frac{dz}{dt} = 1.$$

$\vec{V} = (xy)\hat{i} + (y^2\hat{j}) + e^z\hat{k}$

$\Rightarrow \vec{V}(t) = t^2(2t)\hat{i} + (2t)^2\hat{j} + e^t\hat{k}$

$\Rightarrow \vec{V}(t) = (2t^3\hat{i} + 4t^2\hat{j} + e^t\hat{k})$

Comp. with: $\vec{V}(t) = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$

$\Rightarrow V_1 = 2t^3,\ V_2 = 4t^2,\ V_3 = e^t$; $0 \le t \le 1$

$$\therefore \quad I = \int_C \overline{V} \cdot d\overline{r} = \int_{t=0}^{1} \left(\overline{V(t)} \cdot \frac{d\overline{r}}{dt} \right) dt$$

$$= \int_{t=0}^{1} \left(V_1 \frac{dx}{dt} + V_2 \frac{dy}{dt} + V_3 \frac{dz}{dt} \right) dt$$

$$= \int_{t=0}^{1} \left[(2t^3)(2t) + (4t^2)(2) + (e^t).1 \right] dt$$

$$= \int_{t=0}^{1} (4t^4 + 8t^2 + e^t)\, dt$$

$$= \left[\frac{4t^5}{5} + \frac{8t^3}{3} + e^t \right]_0^1$$

$$= \left[\frac{4}{5} + \frac{8}{3} + e^1 \right] - \left[0 + 0 + e^0 \right]$$

$$\Rightarrow \boxed{I = \frac{37}{15} + e} \quad \text{Ans.}$$

(2). $\overline{V} = x\hat{i} + y\hat{j} + z\hat{k}$, C: Line segment from (1, 2, 2) to (3, 6, 6).

<u>Sol:</u> C : Line segment from (1, 2, 2) to (3, 6, 6).

Equ: of line is:

$$\frac{x-1}{3-1} = \frac{y-2}{6-2} = \frac{z-2}{6-2} = t \quad (\text{say})$$

$$\left[\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} = t \right]$$

$$\Rightarrow \frac{x-1}{2} = \frac{y-2}{4} = \frac{z-2}{4} = t$$

$$\Rightarrow \frac{x-1}{2} = t \quad \Big| \quad \frac{y-2}{4} = t \quad \Big| \quad \frac{z-2}{4} = t$$

$$\Rightarrow x = (2t+1) \quad \Big| \quad y = (4t+2) \quad \Big| \quad z = (4t+2)$$

$x = (2t+1)$ $\quad y = (4t+2)$ $\quad z = (4t+2)$

$\Rightarrow \dfrac{dx}{dt} = 2 \quad \Big| \quad \dfrac{dy}{dt} = 4 \quad \Big| \quad \dfrac{dz}{dt} = 4$

$$\vec{V} = (x\hat{i} + y\hat{j} + z\hat{k})$$

$$\Rightarrow \vec{V}(t) = (2t+1)\hat{i} + (4t+2)\hat{j} + (4t+2)\hat{k} \left\{ \underline{[v_1\hat{i} + v_2\hat{j} + v_3\hat{k}]} \right\}$$

$\therefore$ Reqd. line integral :

$$I = \int_C \vec{V} \cdot d\vec{r} = \int_{t=a}^{b} \left(\vec{V}(t) \cdot \frac{d\vec{r}}{dt} \right) dt$$

Here x varies from 1 to 3 $\quad [(1, 2, 2)$ to $(3, 6, 6)]$

$\therefore$ When $x = 1 \Rightarrow (2t+1) = 1 \Rightarrow t = 0$

When $x = 3 \Rightarrow (2t+1) = 3 \Rightarrow t = 1$

$$\therefore \quad 0 \leq t \leq 1$$

$$\therefore I = \int_{t=0}^{1} \left(v_1 \frac{dx}{dt} + v_2 \frac{dy}{dt} + \frac{dz}{dt} \right) dt$$

$$= \int_{t=0}^{1} \{ (2t+1) \cdot 2 + (4t+2) \cdot 4 + (4t+2) \cdot 4 \} dt$$

$$= \int_{t=0}^{1} (36t + 18)\, dt$$

$$= 18 \left[\cancel{2} \frac{t^2}{\cancel{2}} + t \right]_0^1$$

$$= 18 [(1+1) - 0]$$

$$\Rightarrow \boxed{I = 36} \quad \text{Ans.}$$

Work Done by a Force

Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be a force acting on a particle which moves along a curve C. Then work done by force $\vec{F}$ in displacing the particle from the point P to point Q along the curve C is given by:

$$\boxed{W = \int_P^Q \vec{F}\cdot d\vec{r}} = \int_{t=a}^{b}\left(\vec{F}(t)\cdot\frac{d\vec{r}}{dt}\right)dt$$

$$\Rightarrow \boxed{W = \int_{t=a}^{b}\left(F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt}\right)dt}$$

Find the work done by the force $\vec{F}$ in moving a particle from a point P to point Q.

(1). $\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$, C is the curve represented as: $\vec{r}(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k}$; $1 \le t \le 2$

Sol: Here $\vec{r}(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k}$

$$\Rightarrow x = t,\quad y = t^2,\quad z = t^3$$

$$\Rightarrow \frac{dx}{dt} = 1 \;\Big|\; \frac{dy}{dt} = 2t \;\Big|\; \frac{dz}{dt} = 3t^2$$

$$\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$$

$$\Rightarrow \vec{F} = (t^2)(t^3)\hat{i} + (t^3).(t)\hat{j} + (t).(t^2)\hat{k}$$

$$\Rightarrow \vec{F} = t^5\hat{i} + t^4\hat{j} + t^3\hat{k}$$

$$\Rightarrow F_1 = t^5,\quad F_2 = t^4,\quad F_3 = t^3$$

$\therefore$ Work done, $W = \int_P^Q \vec{F} \cdot d\vec{r}$

$$\Rightarrow W = \int_{t=1}^{2} \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right) dt$$

$$\Rightarrow W = \int_{t=1}^{2} [t^5 (1) + t^4 (2t) + t^3 (3t^2)] dt$$

$$= \int_{t=1}^{2} 6t^5 \, dt = 6 \left[\frac{t^6}{6} \right]_{t=1}^{2}$$

$$= (2^6 - 1^6) = 63$$

$$\Rightarrow \boxed{W = 63} \text{ Ans.}$$

Conservative Vector Field.

A vector field $\vec{F}$ is called conservative (gradient field) if $\boxed{\text{curl } \vec{F} = \vec{0}}$

Show that $\boxed{\vec{F} = yz\hat{i} + xz\hat{j} + xy\hat{k}}$ is conservative.

Here $\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$

$$\Rightarrow \text{curl } \vec{F} = \hat{i} \left[\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(zx) \right] - \hat{j} \left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(yz) \right] + \hat{k} \left[\frac{\partial}{\partial x}(zx) - \frac{\partial}{\partial y}(yz) \right]$$

$$= \hat{i}(x - x) - \hat{j}(y - y) + \hat{k}(z - z)$$

$$= 0\hat{i} - 0\hat{j} + 0\hat{k} = \vec{0}$$

Since $\boxed{\text{curl } \vec{F} = \vec{0}}$ so, $\vec{F}$ is conservative.